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The $q=0$ Quantum Plane

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In this paper we investigate a $q=0$ plane which cannot be obtained by putting $q=0$ in the Wess–Zumino formalism and has no classical analog.

The study of the quantum plane is intended to extend the concept of commutative (or anticommutative) space-time to the more generic case (noncommutative associative space-time) which includes the former two cases as special examples (classical cases). As bosons reside in commutative space-time and fermions reside in anticommutative space-time, a quon[2] is believed to live in q-commutative space-time (quantum plane). Since the development of the geometry of noncommutative space (Manin, 1988, 1989), physicists have studied the differential calculus of this space (Woronowicz, 1987, 1989; Wess and Zumino, 1990; Zumino, 1991; Schmidke *et al.*, 1989; Schirrmacher, 1991a,b; Schirrmacher *et al.*, 1991; Burdik and Hlavaty, 1991; Hlavaty, 1991; Burdik and Hellinger, 1992; Ubriaco, 1992; Giler *et al.*, 1991, 1992; Lukierski *et al.*, 1991; Lukierski and Nowicki, 1992; Castellani, 1992; Chaichian and Demichev, 1992; Chung, n.d.-a,b; Chung *et al.*, 1994). The most pedagogical paper is Wess and Zumino (1990). However, in spite of much effort in this direction, the $q=0$ case has been excluded from the realm of research. Of course, the $q=0$ case is not a deformation, because the deformation parameter does not exist. Moreover, the $q=0$ case has no classical analog. Nevertheless, the $q=0$ oscillator algebra has been shown to be a kind of Hopf algebra (Chung, 1994). Recently Greenberg (1990) discussed the $q=0$ oscillator algebra and showed that the algebra obeys the quantum Boltzmann statistics.

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[2] Here we refer to quons as particles satisfying the q-deformed Heisenberg–Weyl algebra.

In this paper we discuss the $q=0$ plane in a systematic way. It is worth noting that we cannot construct the $q=0$ plane by putting $q=0$ in the Wess–Zumino formalism (Wess and Zumino, 1990). However, we can obtain the differential calculus for the $q=0$ plane and construct an R matrix for the $q=0$ case which satisfies the Yang–Baxter equation but does not have an inverse.

The $q=0$ quantum plane is defined by the two commutation relations[3]

$$xy = 0 \tag{1}$$

$$dx\, dy = 0 \tag{2}$$

which imply that yx and $dy\, dx$ cannot be written in terms of xy and $dx\, dy$, respectively. We want to define an exterior differential d satisfying the usual properties such as

$$d^2 = 0 \tag{3}$$

and the Leibniz rule

$$d(fg) = f\, dg + (df)g \tag{4}$$

where f and g are functions of the variables x and y. Now we obtain the commutation relations among rows and columns for the $q=0$ matrix. Consider the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5}$$

which preserves the relation (1). We assume that entries of this matrix commute with the coordinates x and y. This leads to

$$ac = 0, \qquad bd = 0, \qquad bc = 0 \tag{6}$$

which do not determine all the commutation relations between the entries of the transformation matrix. Let us therefore impose other conditions on the transformation matrix, which are given by equation (2) and the nilpotent property of the exterior differential,

$$dx'\, dy' = 0, \qquad (dx')^2 = (dy')^2 = 0 \tag{7}$$

These result in the following relations:

$$bc = 0, \qquad ba = 0, \qquad dc = 0 \tag{8}$$

The $q=0$ determinant is obtained by the transformation formula

$$dy'\, sx' = D\, dy\, dx \tag{9}$$

[3] Here we put $q=0$ in the three types of solutions given in Brzezinski (1992).

where the $q=0$ determinant D is written as

$$D = da \tag{10}$$

Now let us compute the R matrix for the $q=0$ plane. To do so, we have the following relation:

$$RT_1T_2 = T_2T_1R \tag{11}$$

where

$$T_1 = T \otimes I, \qquad T_2 = I \otimes T \tag{12}$$

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{13}$$

This equation and the relations among the entries of the $q=0$ matrix T give us the following relations:

$$\begin{aligned} R^{11}_{12} &= R^{11}_{21} = R^{11}_{22} = R^{12}_{11} = R^{12}_{22} \\ &= R^{21}_{11} = R^{21}_{21} = R^{21}_{22} = R^{22}_{11} = R^{22}_{12} = R^{22}_{21} = 0 \\ ab(R^{11}_{11} - R^{21}_{12}) &= 0 \\ R^{11}_{11} - R^{22}_{22} &= 0 \\ ca(R^{12}_{21} - R^{11}_{11}) &= 0 \\ R^{12}_{12}(ad - da) &= cb(R^{21}_{12} - R^{12}_{21}) \\ db(R^{12}_{21} - R^{22}_{22}) &= 0 \\ cd(R^{22}_{22} - R^{21}_{12}) &= 0 \end{aligned} \tag{14}$$

Demanding the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{15}$$

and solving the above relations, we obtain the following two solutions[4]

[4] In the appendix we list several solutions of the Yang–Baxter equation. The R matrix (16) and the R' matrix (17) are obtained from R_{III} and R_{I}, respectively, of the appendix.

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{16}$$

$$R' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

where we should impose the relations

$$ad = da, \qquad ab = 0, \qquad cd = 0 \tag{18}$$

In both cases we have no classical analog. In defining the $q=0$ plane, we used the relation $xy = 0$, so we discard the R' solution. Therefore we conclude that there exists an R matrix for the $q=0$ plane satisfying the Yang–Baxter equation even though this type of R matrix does not have an inverse.

Now we discuss the differential calculus for the $q=0$ plane. First we write the commutation relation between coordinates and differentials. The general form, linear in coordinates and consistent with the assumed scale invariance, reads

$$\begin{aligned} x\,dx &= a\,dxx \\ y\,dy &= b\,dyy \\ x\,dy &= c\,dxy + e\,dxy \\ y\,dx &= f\,dxy + g\,dyx \end{aligned} \tag{19}$$

From the consistency condition

$$d(xy) = 0 \tag{20}$$

we get

$$c = 0 \qquad \text{and} \qquad (1 + e)\,dxy = 0 \tag{21}$$

Through the consistency condition

$$xy\,dx = 0 \tag{22}$$

we commute dx through to the left, using equations (19); then we have

$$ge = 0 \tag{23}$$

Similarly, with another consistency condition

$$xy\, dy = 0 \tag{24}$$

we commute dy through to the left, using equations (19); then we have

$$be = 0 \tag{25}$$

Then we have the following two cases.

Case I. $e = -1$ and $dxy \neq 0$. In this case we have $b = g = 0$; then we get $dy\, dx = -f\, dx\, dy = 0$, which leads to a contradiction. So we discard this choice.

Case II. $e \neq -1$ and $dxy = 0$. In this case we have $e = 0$ and $g = -1$. Therefore equations (19) reduce to

$$\begin{aligned} x\, dx &= a\, dxx \\ y\, dy &= b\, dyy \\ x\, dy &= 0 \\ y\, dx &= -dyx \end{aligned} \tag{26}$$

where a and b are not determined at this stage, but will be fixed later by imposing other consistency conditions.

When we act by the differential on the function $f(x, y)$, we use the following definition:

$$df = dx\, \partial_x f + dy\, \partial_y f \tag{27}$$

Substituting $f \to xf$ or $f \to yf$, we find the commutation relations between coordinates and derivatives,

$$\begin{aligned} \partial_x x &= 1 + ax\partial_x \\ \partial_y x &= \partial_y y = 0 \\ \partial_y y &= 1 - x\partial_x + by\partial_y \end{aligned} \tag{28}$$

From the Poincaré lemma for the exterior differentials

$$d^2 = 0 \tag{29}$$

we obtain the commutation relation between derivatives

$$\partial_x \partial_y = 0 \tag{30}$$

Applying ∂_x to the last equation of (28) and using (30) leads to $a = 0$. The next (last) step is to fix the commutation relation between differentials and derivatives and to determine the value of b. To do so we consider the most general form for the commutation relations between differentials and derivatives,

$$
\begin{aligned}
\partial_x\, dx &= A\, dx\, \partial_x + B\, dy\, \partial_y \\
\partial_y\, dx &= C\, dx\, \partial_y + D\, dy\, \partial_x \\
\partial_y\, dy &= E\, dx\, \partial_x + F\, dy\, \partial_y \\
\partial_x\, dy &= G\, dy\, \partial_x + H\, dx\, \partial_y
\end{aligned} \tag{31}
$$

From the relation

$$d(dx) = -dxd, \qquad d(dy) = -dyd \tag{32}$$

we find

$$C = 0 \qquad \text{and} \qquad E = -1 \tag{33}$$

Multiplying the first and second equations of (31) by dy from the right and commuting dy through to the left, we obtain

$$B = 0 \qquad \text{and} \qquad DH = 0 \tag{34}$$

Applying ∂_x to the second and third equations of (31) from the left and commuting ∂_x through to the right, we obtain

$$A = 0 \qquad \text{and} \qquad FH = 0 \tag{35}$$

Multiplying the fourth equation of (31) by x and y from the right and commuting x and y through to the left, we obtain

$$G = 0 \qquad \text{and} \qquad H = 0 \tag{36}$$

Similarly, multiplying the third equation of (31) by x and y from the right and commuting x and y through to the left, we obtain

$$D = 0, \qquad F = -1, \qquad b = -1 \tag{37}$$

Therefore the commutation relations between differentials and derivatives read

$$
\begin{aligned}
\partial_x\, dx &= 0 \\
\partial_y\, dx &= 0 \\
\partial_y\, dy &= -dx\, \partial_x - dy\, \partial_y \\
\partial_x\, dy &= 0
\end{aligned} \tag{38}
$$

Hence we can say that the $q=0$ plane exists and the corresponding R matrix is obtained. We showed that this kind of R matrix satisfies the Yang–Baxter equation although it has no classical analog. We also investigated the differential calculus of the q=0 plane.

APPENDIX. TYPES OF SOLUTIONS FOR YANG–BAXTER EQUATION

In this appendix we introduce some types of solutions of the Yang–Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{A1}$$

$$R_{\mathrm{I}} = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & 0 & k \end{pmatrix} \tag{A2}$$

$$R_{\mathrm{II}} = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & l & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & k \end{pmatrix} \tag{A3}$$

$$R_{\mathrm{III}} = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & l & 0 & 0 \\ 0 & k - lm/k & m & 0 \\ 0 & 0 & 0 & k \end{pmatrix} \tag{A4}$$

$$R_{\mathrm{IV}} = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & l & k - lm/k & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & k \end{pmatrix} \tag{A5}$$

$$R_{\mathrm{V}} = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & \pm k & 0 & 0 \\ 0 & 0 & \pm k & 0 \\ l & 0 & 0 & k \end{pmatrix} \tag{A6}$$

$$R_{\mathrm{VI}} = \begin{pmatrix} k & 0 & 0 & l \\ 0 & \pm k & 0 & 0 \\ 0 & 0 & \pm k & 0 \\ 0 & 0 & 0 & k \end{pmatrix} \tag{A7}$$

$$R_{\mathrm{VII}} = \begin{pmatrix} k & 0 & 0 & l \\ 0 & k & 0 & 0 \\ 0 & 2k & -k & 0 \\ 0 & 0 & 0 & k \end{pmatrix} \tag{A8}$$

$$R_{\mathrm{VIII}} = \begin{pmatrix} k & 0 & 0 & l \\ 0 & -k & 0 & 0 \\ 0 & 2k & k & 0 \\ 0 & 0 & 0 & k \end{pmatrix} \tag{A8}$$

$$R_{\mathrm{IX}} = \begin{pmatrix} k & 0 & 0 & l \\ 0 & k & k & 0 \\ 0 & k & k & 0 \\ k^2/l & 0 & 0 & k \end{pmatrix} \tag{A9}$$

$$R_{\mathrm{X}} = \begin{pmatrix} k & 0 & 0 & l \\ 0 & -k & k & 0 \\ 0 & k & k & 0 \\ -k^2/l & 0 & 0 & k \end{pmatrix} \tag{A10}$$

$$R_{\mathrm{XI}} = \begin{pmatrix} k & 0 & 0 & l \\ 0 & k & k & 0 \\ 0 & k & -k & 0 \\ -k^2/l & 0 & 0 & k \end{pmatrix} \tag{A11}$$

$$R_{\mathrm{XII}} = \begin{pmatrix} k & 0 & 0 & l \\ 0 & -k & k & 0 \\ 0 & k & -k & 0 \\ k^2/l & 0 & 0 & k \end{pmatrix} \tag{A12}$$

$$R_{\mathrm{XIII}} = \begin{pmatrix} k & 0 & 0 & m \\ 0 & \left(\frac{k^2+l^2}{2}\right)^{1/2} & \frac{k+l}{2} & 0 \\ \frac{k+l}{2} & \mathrm{k} & \left(\frac{k^2+l^2}{2}\right)^{1/2} & 0 \\ \frac{1}{m}\left(\frac{k+l}{2}\right) & 0 & 0 & l \end{pmatrix} \tag{A13}$$

$$R_{\mathrm{XIV}} = \begin{pmatrix} k & a & \frac{1-kl}{l^2-kl}a & -\frac{1+l^2-2kl}{l^3-2l^2k+k^2l}a^2 \\ 0 & l & 0 & -a \\ 0 & 0 & l^{-1} & -\frac{1-kl}{l^2-kl}a \\ 0 & 0 & 0 & k \end{pmatrix} \tag{A14}$$

$$R_{\mathrm{XV}} = \begin{pmatrix} k & a & -\frac{1}{kl}a & \frac{1-kl}{l^2k-k^2l}a^2 \\ 0 & l & k-k^{-1} & \frac{1-kl}{kl-k^2}a \\ 0 & 0 & l^{-1} & -\frac{k-k^2l}{l^2k-k^2l}a \\ 0 & 0 & 0 & k \end{pmatrix} \tag{A15}$$

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